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LETTER TO THE EDITOR

On the Lie symmetry approach to Small's equation of nonlinear optics

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Abstract. We have analysed the set of nonlinear equations suggested by Small from the point of view of Lie symmetry. It has been demonstrated that these equations, describing the propagation of an electromagnetic wave in a nonlinear medium, admit various kinds of symmetry transformation for special values of the parameters. It is also shown that the explicit ansatz type solutions of the same equations obtained earlier by Chanda and Ray can be obtained in a very straightforward way from our point transformation analysis. Other solutions are obtained for some special parameter values.

In a recent communication Small (1981) suggested a set of nonlinear partial differential equations for the description of the bunching phenomenon of paraxial rays in a nonlinear medium. These equations were solved in some special situations by Chanda and Ray (1983) with the help of some ansatz type substitutions. However, since it is well known that such a substitutional technique can not exhaust all possible situations, we consider here a methodical procedure for the solution. The method is that of Lie point symmetry (Bluman and Cole 1974). It is quite interesting to observe that the simplest Lie transformation corresponding to constant generators gives the two solutions discussed by Chanda and Ray (1983), with the help of an ansatz type substitution. Apart from these we have also obtained other solutions corresponding to non-trivial symmetry generators.

We start with the formulation. The equations under consideration read

$$\begin{aligned} &a_{xx}+a_{yy}-(\phi_x^2+\phi_y^2-k_0^2)a+\beta a^3=0 \\ &(a^2\phi_x)_x+(a^2\phi_y)^y=0. \end{aligned} \tag{1}$$

Let us consider a set of Lie point transformations given by

$$\begin{aligned} &\phi\to\phi^*=\phi+\varepsilon\eta^1(\phi,a,x,y), \qquad a\to a^*=a+\varepsilon\eta^2(\phi,a,x,y), \\ &x\to x^*=x+\varepsilon\xi_1(\phi,a,x,y), \qquad y\to y^*=y+\varepsilon\xi_2(\phi,a,x,y), \end{aligned} \tag{2}$$

and demand the invariance of (1), that is the equations hold in the same manner even in the new variables:

$$\begin{aligned} &a^*_{x^*x^*}+a^*_{y^*y^*}-(\phi^{*2}_{x^*}+\phi^{*2}_{y^*}-k_0^2)a^*+\beta a^{*3}=0 \\ &(a^{*2}\phi^*_{x^*})_{x^*}+(a^{*2}\phi^*_{y^*})_{y^*}=0. \end{aligned} \tag{3}$$

Substituting from (2) in (3) and considering terms in first order in ε, we get the following equations for the determination of η^i, ξ_i

$$\eta^2_{xx}+\eta^2_{yy}-2a\phi_x\eta^1_x-2a\phi_y\eta^1_y-(\phi_x^2+\phi_y^2-k_0^2)\eta^2+3a^2\beta\eta^2=0 \tag{4}$$

$$\eta^2\phi_{xx}+\eta^2\phi_{yy}+a\eta^1_{xx}+a\eta^1_{yy}+2\phi_x\eta^2_x+2a_x\eta^1_x+2\phi_y\eta^2_y+2a_y\eta^1_y=0. \tag{5}$$

In these equations we denote by η^1_x, η^2_{yy} the total derivative of η^1 or η^2 with respect to x or y. The expressions for these are too long and involved to be quoted here. We only give here the form of the first-order derivatives of η^1 and η^2 (Ovsjannikov 1978)

$$\eta^1_x=\frac{\partial\eta^1}{\partial x}+\frac{\partial\eta^1}{2\phi}\phi_x+\frac{\partial\eta^1}{\partial a}a_x-\frac{\partial\xi_1}{\partial x}\phi_x-\frac{\partial\xi_2}{\partial x}\phi_y$$
$$-\frac{\partial\xi_1}{2\phi}\phi_x^2-\frac{\partial\xi_2}{\partial\phi}\phi_x\phi_y-\frac{\partial\xi_1}{\partial a}a_x\phi_x-\frac{\partial\xi_2}{\partial a}a_x\phi_y \tag{6}$$

$$\eta^2_y=\frac{\partial\eta^2}{\partial y}+\frac{\partial\eta^2}{\partial\phi}\phi_y+\frac{\partial\eta^2}{\partial a}a_y-\frac{\partial\xi_1}{\partial y}a_x-\frac{\partial\xi_2}{\partial y}a_y$$
$$-\frac{\partial\xi_1}{\partial\phi}\phi_y a_x-\frac{\partial\xi_2}{\partial\phi}\phi_y a_y-\frac{\partial\xi_1}{\partial a}a_y a_x-\frac{\partial\xi_2}{\partial a}a_y^2. \tag{7}$$

Substituting these expressions and also inserting those for the second derivatives in (4) and (5), we equate to zero the coefficients of the derivatives of ϕ and a to obtain equations for η^i and ξ_i. The two sets of equations are

$$\begin{aligned}
&\xi_{1\phi}=\xi_{2\phi}-\xi_{1a}=\xi_{2a}=0, \qquad \xi_{1y}=-\xi_{2x}, \qquad \xi_{1x}=\xi_{2y},\\
&\eta^2_{aa}=0, \qquad \eta^2_{\phi\phi}-2a\eta^1_\phi+a\eta^2_a-\eta^2=0, \qquad \eta^2_{\phi a}-a^{-1}\eta^2_\phi-a\eta^1_a=0,\\
&\eta^2_{xa}=0, \qquad \eta^2_{ya}=0, \qquad \eta^2_{x\phi}-a\eta^1_x=0, \qquad \eta^2_{y\phi}-a\eta^1_y=0,\\
&\eta^2_{xx}+\eta^2_{yy}+k_0^2\eta^2+3a^2\beta\eta^2-k_0^2a\eta^2_a-\beta a^2\eta^2_a+2k_0^2\xi_{1x}+2\beta a^3\xi_{1x}=0.
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
&a\eta^1_{aa}+2\eta^1_a=0, \qquad a\eta^1_{\phi\phi}+2\eta^2_\phi+a^2\eta^1_a=0,\\
&a\eta^1_{\phi a}+\eta^2_a-a^{-1}\eta^2=0, \qquad a\eta^1_{xa}+\eta^1_x=0,\\
&a\eta^1_{ya}+\eta^1_y=0, \qquad a\eta^1_{x\phi}+\eta^2_x=0, \qquad a\eta^1_{y\phi}+\eta^2_y=0,\\
&\eta^1_{xx}+\eta^1_{yy}-k_0^2a\eta^1_a-\beta a^3\eta^1_a=0.
\end{aligned} \tag{9}$$

The general forms of η^1 and η^2 are

$$\eta^1=-a^{-1}h(\phi,x,y)+m(\phi), \qquad \eta^2=af(\phi)+g(\phi,x,y) \tag{10}$$

which when inserted in the rest of equations of (8) and (9) suggests the following structures in each individual case.

Case 1. $k_0\neq 0$, $\beta\neq 0$

$$\frac{\partial\xi_1}{\partial y}=-\frac{\partial\xi_2}{\partial x}, \qquad \frac{\partial\xi_1}{\partial x}=\frac{\partial\xi_2}{\partial y}=0, \qquad g=f=h=0, \qquad m=\text{constant}$$

so that

$$\begin{aligned}
&\eta^1=\text{constant}=p\ (\text{say}), \qquad \eta^2=0,\\
&\xi_1=-Ay+B, \qquad \xi_2=Ax+C.
\end{aligned} \tag{11}$$

In the very special situation when $A=0=B$, $C=1$ the Lagrange equations pertaining to this case are;

$$\mathrm{d}x/0=\mathrm{d}y/1=\mathrm{d}\phi/p=\mathrm{d}a/0.$$

So we get

$$\phi=py+f(x), \qquad a=g(x). \tag{12}$$

Substituting in the original equation we get

$$f=D\int \mathrm{d}x/g^2 \tag{13}$$

and

$$g_{xx}-f_x^2 g-(p^2-k_0^2)g+\beta g^3=0. \tag{14}$$

Eliminating f between (13) and (14) we have

$$x=\int \frac{\sqrt{2}g\,\mathrm{d}g}{[-\beta g^6+2(p^2-k_0^2)g^4+Eg^2-2D^2]^{1/2}}. \tag{15}$$

So that we obtain the first set of solutions as

$$\begin{gathered}\phi=D\int \mathrm{d}x/a^2+py \\ x=\int \frac{\sqrt{2}\,a\,\mathrm{d}a}{[-\beta a^6+2(p^2-k_0^2)a^4+Ea^2-2D^2]^{1/2}}.\end{gathered} \tag{16}$$

The second integral in (16) can be equated with the help of elliptic functions. At this point it is worth mentioning that this solution was obtained by Chanda and Ray through an ansatz type calculation, which required some intricate manipulation in the intermediate steps. However, in this paper the same result is obtained in a very simple manner through Lie point symmetry.

Case 2.

$$\eta^1=p \text{ (constant)}, \qquad \eta^2=0, \qquad \xi_1=-y, \qquad \xi_2=x.$$

The corresponding Lagrange equations are

$$\mathrm{d}x/-y=\mathrm{d}y/x=\mathrm{d}\phi/p=\mathrm{d}a/0. \tag{17}$$

Integration of these leads to

$$\begin{gathered}x^2+y^2=\text{constant}=r^2 \\ \phi=p\sin^{-1}(y/r)+f(r).\end{gathered} \tag{18}$$

Transforming to polar coordinate in two dimensions (i.e. $x=r\cos\theta$, $y=r\sin\theta$) we get

$$\phi=f(r)+p\theta, \qquad a=g(r). \tag{19}$$

Substituting in the original equation transformed to polar variables we get

$$g_r/r+g_{rr}-(f_r^2+p^2/r^2-k_0^2)g+\beta g^3=0 \tag{20}$$

and

$$f = \int A\, \mathrm{d}r/rg^2 + B. \tag{21}$$

Eliminating f we obtain

$$r(rg_r)_r = A^2/g^3 + r^2[g(p^2/r^2 - k_0^2) - \beta g^3], \tag{22}$$

an ordinary nonlinear differential equation for g.

At this point it is worth mentioning that no further symmetry transformation is seen to exist for the case $\beta \neq 0$, $k_0 \neq 0$ as in the presence of the term βa^3 our equation behaves in a manner similar to the nonlinear Klein-Gordon system $\phi_{xx} - \phi_{tt} = \lambda\phi^3$, which is also known to possess only two or three symmetry transformations. However again it is interesting to note that our equation (22) is identical to equation (14) of Chanda and Ray. So we have again reproduced another result of this ansatz type calculation of the solution. To explore the other type of solutions we now consider the situation when β is small and we can neglect the term βa^3. Note that even after neglecting β these equations remain nonlinear. So we obtain the following.

Case 3. $\beta = 0$, $k_0 \neq 0$.

$$\begin{gathered}
\frac{\partial \xi_1}{\partial y} = -\frac{\partial \xi_2}{\partial x}, \qquad \frac{\partial \xi_1}{\partial x} = \frac{\partial \xi_2}{\partial y} = 0, \qquad \frac{\partial g}{\partial \phi} + h = 0, \qquad \frac{\partial h}{\partial \phi} - g = 0 \\
\frac{\partial^2 f}{\partial \phi^2} - 2\frac{\partial m}{\partial \phi} = 0, \qquad \frac{\partial^2 m}{\partial \phi^2} + 2\frac{\partial f}{\partial \phi} = 0 \\
\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + k_0^2 g = 0, \qquad \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + k_0^2 h = 0
\end{gathered} \tag{23}$$

whose solutions are

$$\begin{aligned}
\eta^1 &= -a^{-1}[p(x, y)\sin\phi + q(x, y)\cos\phi] + (A\sin 2\phi + B\cos 2\phi) \\
\eta^2 &= a[A\sin 2\phi + B\cos 2\phi] + [p(x, y)\cos\phi + q(x, y)\sin\phi] \\
\xi_1 &= -A'y + B', \qquad \xi_2 = A'x + C'
\end{aligned}$$

with

$$p_{xx} + p_{yy} + k_0^2 p = 0, \qquad q_{xx} + q_{yy} + k_0^2 q = 0. \tag{24}$$

Since it is impossible to integrate the Lagrange equations with these η^1, η^2, ξ_1, ξ_2 we consider a special case of (24) with $p = q = 0$. In this case the Lagrange equations are

$$\frac{\mathrm{d}x}{-y} = \frac{\mathrm{d}y}{x} = \frac{\mathrm{d}\phi}{A\sin 2\phi + B\cos 2\phi} = \frac{\mathrm{d}a}{a(-A\cos 2\phi + B\sin 2\phi)}. \tag{25}$$

By integration of $x^2 + y^2 = r^2$

$$\frac{\mathrm{d}y}{(r^2 - y^2)^{1/2}} = \frac{\mathrm{d}\phi}{A\sin 2\phi + B\cos 2\phi}$$

which yields

$$\phi = \tan^{-1}\exp[2(A^2 + B^2)^{1/2}(f(r) \pm \theta)] - \alpha. \tag{26}$$

The other pair of (25) gives

$$a = g \operatorname{cosec}^{1/2} 2(\phi + \alpha). \tag{27}$$

Substituting in the original equations

$$f_r / r + f_{rr} + 2 f_r g_r / g = 0 \tag{28}$$

and

$$r^{-1}(\partial/\partial r)(rg) + (A^2 + B^2)(c^2/r^2 g^3 + g/r^2) + k_0^2 g = 0. \tag{29}$$

Equation (29) in the special case of $c = 0$ reduces to a Bessel equation and can be solved explicitly. Lastly we mention another case in which the Lagrange equations can be explicitly integrated.

Case 4.

$$\begin{aligned} \eta^1 &= A \sin 2\phi + B \cos 2\phi, \\ \eta^2 &= a(-A \cos 2\phi + B \sin 2\phi), \\ \xi_1 &= 0, \qquad \xi_2 = 1. \end{aligned} \tag{30}$$

In this case we get

$$\phi = \tan^{-1} \exp[2(A^2 + B^2)^{1/2}(f(x) + y)] - \alpha$$

$$a = g(x) \operatorname{cosec}^{1/2} 2(\phi + \alpha).$$

Inserting these equations in the original equations we get

$$g\, \mathrm{d}g/\mathrm{d}x = [2M^2(A^2 + B^2) - \lambda g^4 + 2Ng^2]^{1/2}$$

or

$$\mathrm{d}h/\mathrm{d}x = [2M^2(A^2 + B^2) - \lambda h^2 + 2Nh]^{1/2}$$

or

$$\int \frac{\mathrm{d}h}{[2M^2(A^2 + B^2) - \lambda h^2 + 2Nh]^{1/2}} = x + c_0 \tag{31}$$

which can be easily evaluated in terms of elementary functions. Also f is given as

$$f = \int M\, \mathrm{d}x/g^2. \tag{32}$$

In the above discussion we have shown how the application of Lie point symmetry to the equations of nonlinear optics of Small yields the different symmetry transformations of the equation and simultaneously generates some physical solutions, previously obtained by ansatz type substitutions.

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